

# The Vacuum Structure of Light-Front $\phi^4_{1+1}$ -Theory

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## Abstract

We discuss the vacuum structure of  $\phi^4$ -theory in 1+1 dimensions quantised on the light-front  $x^+ = 0$ . To this end, one has to solve a non-linear, operator-valued constraint equation. It expresses that mode of the field operator having longitudinal light-front momentum equal to zero, as a function of all the other modes in the theory. We analyse whether this zero mode can lead to a non-vanishing vacuum expectation value of the field  $\phi$  and thus to spontaneous symmetry breaking. In perturbation theory, we get no symmetry breaking. If we solve the constraint, however, non-perturbatively, within a mean-field type Fock ansatz, the situation changes: while the vacuum state itself remains trivial, we find a non-vanishing vacuum expectation value above a critical coupling. Exactly the same result is obtained within a light-front Tamm-Dancoff approximation, if the renormalisation is done in the correct way.

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# 1 Introduction

Back in 1949, Dirac [1] noted that within a relativistic formulation of Hamiltonian dynamics the choice of the time evolution parameter  $\tau$  is not unique. As an alternative to the usual cartesian time  $\tau = t$  of Galileian dynamics he suggested the variable  $\tau = x^+ \equiv t + z/c$ , now called “light-cone time”. For this choice, hyperplanes tangent to the light-cone, *i.e.* light-fronts or null-planes,  $x^+ = \text{const}$ , are surfaces of equal time. The associated Hamiltonian  $H$  then describes the time evolution in  $x^+$  off these surfaces.

Independently, in the early sixties, the idea of specifying initial data on null-hyper-surfaces was developed by Penrose and others with particular regard to the gravitational field [2].

It was not before the late sixties, however, when particle physicists became aware of Dirac’s work. It was realized that the infinite-momentum limit used in current algebra [3] amounts to using “light-like” charges defined as integrals over null-planes [4]. Soon after this observation, quantisation of field theories on light-fronts was explicitly formulated [5, 6, 7]. The first reference to Dirac was made in a paper by Chang and Ma [8] on light-front (LF) perturbation theory. For reviews on the early achievements of LF field theory see [9].

Already at that time it was repeatedly stated that one of the main advantages of the infinite-momentum frame (and hence light-front quantisation) is the simplicity of the vacuum structure. As early as 1966, Weinberg realized that within “old-fashioned” Hamiltonian perturbation theory the infinite-momentum limit of many diagrams, in particular vacuum diagrams, is vanishing [10]. The success of this limit for current algebra was traced in [6] to the fact that light-like charges always annihilate the vacuum, irrespective of whether they are conserved or not. Thus, Coleman’s theorem [11], “the symmetry of the vacuum is the symmetry of the world”, does not apply. Furthermore, the Fock-vacuum, *i.e.* the ground state of the free Hamiltonian, is stable under interaction. In the same manner as the light-like charges, the fully interacting Hamiltonian annihilates the vacuum, at least in theories with a mass gap [6]. The technical reason for this is the positivity of a kinematical Poincaré generator, the longitudinal momentum  $P^+ = P^0 + P^3$ : as any (massive) particle carries positive  $p^+$  it cannot be degenerate with the vacuum having  $p^+ = 0$ . These results were then transformed into the folkloristic statement: “on the light-cone, the vacuum is trivial”.

With the advent of QCD as the theory of strong interactions, however, people began to feel uneasy with this statement. There was (and is) growing evidence, that many of the phenomenological aspects of hadron physics, like confinement and chiral symmetry

breaking are related to the non-trivial features of the QCD vacuum (within standard equal-time quantisation on a space-like hypersurface). Let us only mention features like quark- and gluon condensates, instantons, monopole condensation etc., which all indicate that the vacuum is densely populated by non-trivial quantum fluctuations, which furthermore are not accessible to perturbation theory.

The concern that arose at these points can be put into the question: can the existence of these large vacuum fluctuations be reconciled with the triviality of the light-front vacuum? For QCD, the answer to this question is not (yet) known. For simpler theories, it depends to some extent on the theory. Generally, one can say the following. Not unexpectedly, the delicate point is the behaviour of the degrees of freedom at longitudinal momentum  $p^+ = 0$ . At this point, even the energy<sup>1</sup>

$$p^- = \frac{p_\perp^2 + m^2}{2p^+} \quad (1.1)$$

of a free particle of mass  $m$  diverges. In perturbation theory, one encounters associated infrared divergences. These are conveniently regularised by working in a finite spatial volume. In this way, the modes having  $p^+ = 0$ , shortly called zero modes (ZMs), can be explicitly isolated and studied [12 – 28]. It is generally believed that these modes carry the information on the non-trivial vacuum aspects. This has been shown in particular for  $\phi^4$ -theory in  $d = 1 + 1$ , where the ZMs are responsible for spontaneous symmetry breaking [18, 21, 23, 24, 25]. This article elaborates on the quoted works on  $\phi_{1+1}^4$ -theory by extending the approximations made there and clarifying subtleties of the renormalisation procedure. The different approaches are compared in detail.

This paper is organized as follows. In Section 2 we shortly review the canonical phase space structure of LF field theories which generically display constraints. For LF  $\phi_{1+1}^4$ , it is the vacuum expectation value of the field  $\phi$  that is constrained. This constraint is solved by a perturbative expansion in Section 3, by a mean-field Fock ansatz in Section 4 and within a LF Tamm-Dancoff approximation in Section 5.

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<sup>1</sup>our LF conventions are:  $x^\pm = (x^0 \pm x^3)/\sqrt{2}$ ,  $\partial_\pm = \partial/\partial x^\pm$ .

## 2 Light-Front Field Theory as a First Order System

The Lagrangian of LF scalar field theory in 1+1 dimensions

$$L[\phi, \partial_+ \phi] = \int dx^- \partial_- \phi \partial_+ \phi - U[\phi] \quad (2.1)$$

is *linear* (*i.e.* first order) in the LF velocity  $\partial_+ \phi$ . The quantisation of such systems is not quite straightforward. People commonly refer to Dirac's treatment of constrained dynamics [29, 30] with its categorizing of constraints into primary, secondary, ..., first-class and second-class. For LF field theories this was first employed by Banyai, Mezincescu [31] and others [12, 30, 32]. There is, however, a much more economic method for first-order systems due to Faddeev and Jackiw [33], which was to some extent anticipated in [34]. In the context of LF field theory it has only been used in a few recent publications [35, 36]. We shortly review the method for a finite number of degrees of freedom and concentrate on those issues which will be relevant for LF field theory.

Consider a Lagrangian

$$L(x) = \frac{1}{2} x_i f_{ij} \dot{x}_j - \Phi(x) ; \quad i, j = 1, \dots, N , \quad (2.2)$$

where the matrix  $f_{ij}$  is antisymmetric. If it has an inverse,  $f_{ij}^{-1}$ , the canonical bracket between the  $x$ -variables (generalizing the well-known Poisson bracket) is

$$\{x_i, x_j\} = f_{ij}^{-1} . \quad (2.3)$$

In this case the number  $N$  of  $x$ 's must be even, and one can introduce canonical coordinates  $q_\alpha$  and momenta  $\pi_\alpha$  (*i.e.* a polarization and Poisson brackets) as discussed extensively in Ref. [37]. If the matrix  $f_{ij}$  does not have an inverse, there are zero modes  $z^a$ , satisfying

$$f_{ij} z_j^a = 0 . \quad (2.4)$$

The Lagrangian (2.2) can then be cast into the form

$$L(y, z) = \frac{1}{2} y_m \hat{f}_{mn} \dot{y}_n - H(y, z) ; \quad m, n = 1, \dots, N' , \quad (2.5)$$

where the matrix  $\hat{f}_{mn}$  is the invertible sub-block of  $f_{ij}$ , thus  $N' < N$  and the number  $N'$  of  $y$ 's is even.  $H$  denotes the Hamiltonian. The  $z$ -variables are constrained via their equation of motion,

$$\frac{\partial H}{\partial z^a} = 0 , \quad (2.6)$$

and, as stressed by Faddeev and Jackiw, these are the only true constraints in the theory. They should be used to eliminate the  $z$ 's, which might turn out to be difficult or impossible<sup>2</sup>. The  $y$ 's are the unconstrained, “true” degrees of freedom. They have the canonical bracket

$$\{y_m, y_n\} = \hat{f}_{mn}^{-1} ; \quad m, n = 1, \dots, N' . \quad (2.7)$$

Let us extend this discussion to field theory. The main new problem arising is of course the fact that the number of degrees of freedom becomes infinite. Matrices therefore become differential operators. If one looks at the Lagrangian (2.1), one readily notes that the matrix  $f_{ij}$  is replaced by the spatial derivative  $\partial_- = \partial/\partial x^-$ . In order to uniquely define its inverse one has to specify its domain and boundary conditions. We therefore enclose our spatial variable  $x^-$  in a box,  $-L \leq x^- \leq L$ , and impose periodic boundary conditions (pBC) on our fields. This is, of course, nothing but an infrared regularisation. In this case, the operator  $\partial_-$  has zero modes, namely all spatially constant functions.

If we split our field  $\phi$  into a ZM  $\omega$  and its complement  $\varphi$ ,

$$\phi(x^+, x^-) = \omega(x^+) + \varphi(x^+, x^-) , \quad (2.8)$$

$$\omega(x^+) \equiv \frac{1}{2L} \int_{-L}^L dx^- \phi(x^+, x^-) , \quad (2.9)$$

such that

$$\int_{-L}^L dx^- \varphi(x^+, x^-) = 0 , \quad (2.10)$$

the Lagrangian (2.1) can be rewritten analogous to (2.5),

$$L[\varphi, \omega] = \int_{-L}^L dx^- \frac{1}{2} \varphi(-2\partial_-) \dot{\varphi} - H[\varphi, \omega] . \quad (2.11)$$

The Hamiltonian  $H$  is identical with the potential  $U$  from (2.1) after the replacement (2.8) has been performed. Thus we see that the ZM  $\omega$  is the analogue of the  $z$ -variables and therefore constrained via

$$\frac{\delta H}{\delta \omega} = 0 . \quad (2.12)$$

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<sup>2</sup>The particular case when  $H$  is linear in (some of) the  $z$ 's will not be considered here. It leads to additional constraints between the  $y$ -variables and is typical for gauge theories, where these constraints correspond to Gauss's law[33].

The basic bracket is given by

$$\{\varphi(x^+, x^-), \varphi(x^+, y^-)\} = -\frac{1}{2}\langle x^- | \partial_-^{-1} | y^- \rangle , \quad (2.13)$$

where the matrix element on the *r.h.s.* denotes the periodic sign function[36]

$$\langle x^- | \partial_-^{-1} | y^- \rangle \equiv \frac{1}{2} \text{sgn}(x^- - y^-) - \frac{x^- - y^-}{2L} . \quad (2.14)$$

The discussion above becomes especially transparent if one goes to momentum space. Expanding the field into Fourier modes

$$\phi = a_0 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi n}} (a_n e^{-i\pi n x^- / L} + a_n^* e^{i\pi n x^- / L}) , \quad (2.15)$$

the Lagrangian becomes (discarding a total time derivative [33])

$$L(a_n, a_0) = -i \sum_{n>0} a_n \dot{a}_n^* - H(a_n, a_0) , \quad (2.16)$$

where  $a_0 \equiv \omega$  is the constrained ZM. If we introduce a momentum cutoff  $N$ ,  $n < N$ , we have mapped the field theory on a finite dimensional system. The elementary bracket between the Fourier modes can be read off from the kinetic term,

$$\{a_m, a_n^*\} = -i\delta_{mn} . \quad (2.17)$$

Quantisation is performed by employing the correspondence principle, *i.e.* by replacing  $i$  times the canonical bracket by the commutator. For arbitrary classical observables,  $A$ ,  $B$ , this means

$$[\hat{A}, \hat{B}] = i\{\widehat{A}, \widehat{B}\} , \quad (2.18)$$

so that, from (2.17), our elementary commutator becomes

$$[a_m, a_n^\dagger] = \delta_{mn} . \quad (2.19)$$

As is well known, not all classical observables  $A$ ,  $B$  can be quantised unambiguously due to possible operator ordering problems. Such problems do not arise for the field  $\varphi$  and the bracket (2.13), where the field-independent *r.h.s.* leads to a *c*-number commutator. The constraint (2.12), however, implies a functional dependence of the ZM  $\omega$  on  $\varphi$  and thus a non-vanishing commutator of  $\omega$  with  $\varphi$ . This can be explicitly verified by calculating the associated Dirac bracket within the Dirac-Bergmann algorithm [16, 17]. For the quantum theory, this results in an ordering ambiguity with respect to  $\omega$  and  $\varphi$ . Therefore, a definite ordering has to be prescribed. We chose Weyl (or symmetric) ordering [18, 24] which is

explicitly hermitian. Using this prescription, one finds the quantum Hamiltonian for LF  $\phi_{1+1}^4$ -theory,

$$\begin{aligned}
H = & \int_{-L}^L dx^- \left( \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right) + \\
& + \int_{-L}^L dx^- \left[ \frac{1}{2} m^2 \omega^2 + \frac{\lambda}{4!} (\omega^4 + \omega \varphi^3 + \varphi \omega \varphi^2 + \varphi^2 \omega \varphi + \varphi^3 \omega + \right. \\
& \left. + \omega^2 \varphi^2 + \varphi^2 \omega^2 + \omega \varphi \omega \varphi + \varphi \omega \varphi \omega + \omega \varphi^2 \omega + \varphi \omega^2 \varphi) \right]. \quad (2.20)
\end{aligned}$$

Note that we have chosen the sign of the mass term(s) in such a way that there is no spontaneous symmetry breaking at tree level. With this Hamiltonian, equation (2.12) for the constrained ZM  $\omega$  reads explicitly

$$\theta = \frac{\delta H}{\delta \omega} \equiv m^2 \omega + \frac{\lambda}{3!} \omega^3 + \frac{\lambda}{3!} \frac{1}{2L} \int_{-L}^L dx^- [\varphi^3 + \varphi^2 \omega + \varphi \omega \varphi + \omega \varphi^2] = 0. \quad (2.21)$$

This is nothing but the ZM of the Euler-Lagrange equation of motion for the total field  $\phi$  decomposed into  $\omega$  and  $\varphi$  [12, 14, 24]. The remainder of this paper is concerned with different approaches to solve this equation for  $\omega$ .

### 3 Perturbative Solution

To obtain a perturbative solution for  $\omega$  we expand it in a power series in  $\lambda$ ,

$$\omega \equiv \sum_{n=0}^{\infty} \lambda^n \omega_n \quad (3.1)$$

Inserting this into (2.21) determines the coefficients  $\omega_n$  recursively. For the first three we find

$$\omega_0 = 0, \quad (3.2)$$

$$\omega_1 = -\frac{1}{6m^2} \frac{1}{2L} \int_{-L}^L dx^- \varphi^3(x), \quad (3.3)$$

$$\omega_2 = \frac{1}{36m^4} \frac{1}{(2L)^2} \int_{-L}^L dx^- dy^- [\varphi^2(x) \varphi^3(y) + \varphi(x) \varphi^3(y) \varphi(x) + \varphi^3(y) \varphi^2(x)] . \quad (3.4)$$

All higher orders may be obtained similarly. Unfortunately, however, we have not been able to find a closed formula for  $\omega_n$  in order to sum up the whole series (3.1).

If we expand the quantum field  $\varphi$  according to (2.15) in terms of Fock operators,

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi n}} \left[ a_n e^{-ik_n^+ x^-} + a_n^\dagger e^{ik_n^+ x^-} \right], \quad (3.5)$$

with the discretised longitudinal momentum  $k_n^+ = \pi n/L$ , one notes that the vacuum expectation value (VEV) of  $\omega$  is zero to all orders in  $\lambda$ , since  $\omega_n$  contains an *odd* number of Fock operators, *i.e.*  $\langle 0|\omega_n|0\rangle = 0$ . Thus

$$\langle 0|\omega|0\rangle = \sum_{n=0}^{\infty} \lambda^n \langle 0|\omega_n|0\rangle = 0. \quad (3.6)$$

This seems to imply that a non-vanishing VEV for  $\omega$  can only arise non-perturbatively and must be non-analytic in the coupling  $\lambda$ . We will discuss this issue in the next sections and continue for the time being within the framework of perturbation theory. In particular, we will study the effect of the ZM  $\omega$  on the mass renormalisation.

To this end, we split up the Hamiltonian  $H$  from (2.20) into two pieces,

$$H = H_0 + H_\omega, \quad (3.7)$$

where  $H_0$  is independent of  $\omega$ ,

$$H_0 = \int_{-L}^L dx^- \left( \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right), \quad (3.8)$$

and  $H_\omega$  is the  $\omega$ -dependent interaction,

$$\begin{aligned} H_\omega \equiv \int_{-L}^L dx^- \left[ \frac{1}{2} m^2 \omega^2 + \frac{\lambda}{4!} \left( \omega^4 + \omega \varphi^3 + \varphi \omega \varphi^2 + \varphi^2 \omega \varphi + \varphi^3 \omega + \right. \right. \\ \left. \left. + \omega^2 \varphi^2 + \varphi^2 \omega^2 + \omega \varphi \omega \varphi + \varphi \omega \varphi \omega + \omega \varphi^2 \omega + \varphi \omega^2 \varphi \right) \right]. \end{aligned} \quad (3.9)$$

This can be simplified using the constraint equation (2.21)

$$\begin{aligned} H_\omega &= H_\omega - \frac{L}{2} (\omega \theta + \theta \omega) \\ &= \frac{\lambda}{4!} \int_{-L}^L dx^- \left( -\omega^4 + \varphi \omega \varphi^2 + \varphi^2 \omega \varphi + \varphi \omega^2 \varphi - \omega \varphi^2 \omega \right). \end{aligned} \quad (3.10)$$

Because  $\omega$  is of order  $\lambda$ , the ZM dependent part  $H_\omega$  is of order  $\lambda^2$ . Explicitly, we find



$$\begin{aligned}
H_\omega &= \frac{\lambda}{4!} \int_{-L}^L dx^- \left( \varphi \omega \varphi^2 + \varphi^2 \omega \varphi \right) + O(\lambda^3) \\
&= -\frac{\lambda^2}{144m^2} \frac{1}{2L} \int_{-L}^L dx^- dy^- \left[ \varphi(x) \varphi^3(y) \varphi^2(x) + \varphi^2(x) \varphi^3(y) \varphi(x) \right] + O(\lambda^3), \quad (3.11)
\end{aligned}$$

where we have used the first order term (3.3). This induces a mass shift of order  $\lambda^2$  which is given by [17] (see also [21])

$$\delta m^2 \equiv \frac{2\pi n}{L} \left( \langle n | H_\omega | n \rangle - \langle 0 | H_\omega | 0 \rangle \right). \quad (3.12)$$

Here,  $|n\rangle \equiv a_n^\dagger |0\rangle$  denotes a one-particle state of longitudinal momentum  $k_n^+ = \pi n/L$ . We have subtracted the constant vacuum energy,  $\langle 0 | H_\omega | 0 \rangle$ , which diverges linearly with the volume. This is sufficient to render the mass shift finite. After inserting the Fock-expansion (3.5) into (3.11) one obtains for (3.12)

$$\begin{aligned}
\delta m^2 &= -\frac{\lambda^2}{6m^2} \frac{L}{(4\pi)^3} \frac{1}{n} \left[ \sum_{m=1}^{n-1} \frac{1}{(n-m)m} + 4 \sum_{m=1}^{\infty} \frac{1}{(n+m)m} \right] \frac{2\pi n}{L} \\
&= -\frac{\lambda^2}{6m^2} \frac{L}{(4\pi)^3} \frac{2}{n^2} [3\gamma + \Psi(n) + 2\Psi(1+n)] \frac{2\pi n}{L} < 0, \quad (3.13)
\end{aligned}$$

where  $\gamma$  is Euler's constant and  $\Psi$  denotes the Digamma-function [38]. The question now is, whether the result (3.13) is a finite size effect, *i.e.* vanishing in the infinite volume limit. The latter is obtained by replacing

$$\sum_{n=1}^N f(n) \rightarrow \lim_{L \rightarrow \infty} \frac{L}{\pi} \int_{\pi/L}^{\pi N/L} dk^+ f(k^+). \quad (3.14)$$

in such a way that  $k_n^+$  approaches a *finite* limit  $k^+$ . For the case at hand, this amounts to replacing the Digamma-function by its asymptotics [38]

$$\Psi(z) \simeq \ln(z) - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} + \dots \quad (z \rightarrow \infty \quad \text{in} \quad |\arg(z)| < \pi). \quad (3.15)$$

Thus we finally obtain

$$\lim_{L \rightarrow \infty} \delta m^2 = -\frac{\lambda^2}{m^2} \frac{1}{16\pi k^+} \lim_{L \rightarrow \infty} \frac{1}{L} \left[ \gamma + 3 \ln(k^+ L/\pi) + \dots \right] = 0. \quad (3.16)$$

The vanishing of this expression implies that the ZM induced second order mass shift  $\delta m^2$  is indeed a finite size effect. We do not have a general proof that this is also true for higher orders in  $\lambda$ . However, it seems to be plausible that a *single* mode like  $\omega$  constitutes a “set of measure zero” within the infinite number of modes as long as one

applies perturbation theory. From the theory of condensation, however, it is well known that single modes, especially those with vanishing momentum, can significantly alter the perturbative results. To analyse this possibility one clearly has to use non-perturbative methods. This will be done in the next sections.

## 4 Mean-Field Ansatz

As  $\phi_{1+1}^4$ -theory is super-renormalisable, the number of divergent diagrams is finite, namely one: the tadpole resulting from a self-contraction of the field at the same space-time point. Therefore, renormalisation can, at least within perturbation theory, be done via normal-ordering, which is nothing but making use of Wick's theorem: expanding the appearing powers of  $\phi$  in a sum of normal-ordered terms with more and more self-contractions, one separates the convergent term (with no contraction) from the divergent ones (with at least one contraction). The latter terms then are just the negative of the required counterterms. For conventional  $\phi^4$ -theory one finds

$$\begin{aligned} \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 &= \frac{1}{2}m^2\left(:\phi^2: + T\right) + \frac{\lambda}{4!}\left(:\phi^4: + 6T:\phi^2: + 3T^2\right) \\ &= \frac{1}{2}\left(m^2 + \frac{\lambda}{2}T\right): \phi^2: + \frac{\lambda}{4!}:\phi^4: + \frac{m^2}{2}T + \frac{\lambda}{8}T^2, \end{aligned} \quad (4.1)$$

where

$$T \equiv \langle \phi^2 \rangle = \int \frac{dk}{4\pi} \frac{1}{\sqrt{k^2 + m^2}} \quad (4.2)$$

formally denotes the logarithmically divergent tadpole contribution (in  $d = 1+1$ ) which coincides with the VEV of  $\phi^2$  or a self-contraction of the field.

It is obvious that only mass and vacuum energy get renormalised. The renormalised Hamiltonian is obtained by adding the counterterms

$$\delta\mathcal{H} \equiv -\frac{\lambda}{4}T : \phi^2 : - \frac{1}{2}m^2T - \frac{\lambda}{8}T^2, \quad (4.3)$$

If the VEV of  $\phi^2$  is taken in a Fock vacuum corresponding to the bare mass  $m$ , the latter coincides with the renormalised mass to order  $\lambda$ .

This rather trivial renormalisation procedure cannot be straightforwardly extended to LF field theory, simply because we do not know the Hamiltonian! The ZM  $\omega$  is a complicated functional of the Fock operators  $a_n, a_n^\dagger$ , which has to be found from the

constraint (2.21) before we can normal-order. There is, however, a way around this obstacle, such that an *exact* knowledge of  $\omega$  is not needed for renormalisation.

To this end we use the following general ansatz for  $\omega$ :

$$\begin{aligned}
\omega = & \omega_0 + \sum_{n>0} \omega_n a_n^\dagger a_n + \sum_{m,n>0} \omega_{mn} a_m^\dagger a_n^\dagger a_{m+n} + \sum_{m,n>0} \omega_{mn}^* a_{m+n}^\dagger a_m a_n + \\
& + \sum_{l,m,n>0} \omega_{lmn} a_{l+m+n}^\dagger a_l a_m a_n + \sum_{l,m,n>0} \omega_{lmn}^* a_l^\dagger a_m^\dagger a_n^\dagger a_{l+m+n} + \\
& + \sum_{k,l,m,n>0} \delta_{k+l,m+n} \omega_{klmn} a_k^\dagger a_l^\dagger a_m a_n + \dots .
\end{aligned} \tag{4.4}$$

This ansatz is hermitian and takes care of the fact that  $\omega$  cannot transfer any momentum. It can be understood as a Wick expansion written in the opposite of the usual order: the first term  $\omega_0$  is the sum of all contractions, the second the sum of all contractions but one and so on. Accordingly, each individual term in the expansion is a normal-ordered operator. In [18] this ansatz was used (with a truncation after the second term) to determine  $\omega$  and the vacuum structure of the theory. In the following we will analyse the renormalisation structure in more detail and extend the analysis to the calculation of one-particle energies and the ZM induced mass shift.

Inserting this ansatz into (2.20) and (2.21) we obtain for the constraint and the Hamiltonian

$$\theta = \theta_0 + \sum_{n>0} \theta_n a_n^\dagger a_n + \dots , \tag{4.5}$$

$$H = H_0 + \sum_{n>0} H_n a_n^\dagger a_n + \dots , \tag{4.6}$$

where, in accordance with the truncation of our ansatz (4.4), we have omitted terms containing more than two Fock operators. The coefficients  $H_0$ ,  $H_n$  and  $\theta_0$ ,  $\theta_n$  are functions of  $\omega_0$  and  $\omega_n$ . Thus, (4.6) is an effective one-body or mean-field (MF) Hamiltonian describing the influence of the ZM  $\omega$ . Explicitly, one finds for the coefficients of the constraint

$$\theta_0 = \left(m^2 + \frac{\lambda}{2}T\right) \omega_0 + \frac{\lambda}{3!} \left(\omega_0^3 + \sum_{n>0} \frac{\omega_n}{4\pi n}\right) , \tag{4.7}$$

$$\theta_n = \left(m^2 + \frac{\lambda}{2}T\right) \omega_n + \frac{\lambda}{3!} \left(\omega_n^3 + 3\omega_0 \omega_n^2 + 3\omega_0^2 \omega_n + \frac{6\omega_0}{4\pi n} + \frac{6\omega_n}{4\pi n}\right) , \tag{4.8}$$

and of the Hamiltonian (scaled by  $2L$ )

$$\frac{H_0}{2L} = \frac{1}{2} \left( m^2 + \frac{\lambda}{2} T \right) \omega_0^2 + \frac{\lambda}{4!} \left( \omega_0^4 + 4 \sum_{n>0} \frac{\omega_0 \omega_n}{4\pi n} + \sum_{n>0} \frac{\omega_n^2}{4\pi n} \right) + \frac{m^2}{2} T + \frac{\lambda}{8} T^2, \quad (4.9)$$

$$\begin{aligned} \frac{H_n}{2L} &= \frac{1}{2} \left( m^2 + \frac{\lambda}{2} T \right) \left( \omega_n^2 + 2\omega_0 \omega_n + \frac{1}{2\pi n} \right) + \\ &+ \frac{\lambda}{4!} \left( (\omega_0 + \omega_n)^4 - \omega_0^4 + \frac{3}{\pi n} (\omega_0 + \omega_n)^2 + \frac{\omega_n^2}{2\pi n} \omega_n \sum_{m>0} \frac{\omega_m}{\pi m} \right). \end{aligned} \quad (4.10)$$

$T$  now denotes the LF tadpole (in discretised form)

$$T = \langle \varphi^2 \rangle = \sum_{n>0} \frac{1}{4\pi n}, \quad (4.11)$$

which is mass independent in contrast to (4.2). Note that the one-particle matrix elements of  $\theta$  and  $H$  are given as the *sum* of two coefficients

$$\begin{aligned} \langle n | \theta | n \rangle &= \theta_0 + \theta_n = \\ &= \left( m^2 + \frac{\lambda}{2} T \right) (\omega_0 + \omega_n) + \frac{\lambda}{3!} \left[ (\omega_0 + \omega_n)^3 + 6 \frac{\omega_0 + \omega_n}{4\pi n} + \sum_{k>0} \frac{\omega_k}{4\pi k} \right], \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} \langle n | H/2L | n \rangle &= (H_0 + H_n)/2L = \\ &= \frac{1}{2} \left( m^2 + \frac{\lambda}{2} T \right) (\omega_0 + \omega_n)^2 + \frac{1}{2} \left( m^2 + \frac{\lambda}{2} T \right) \frac{1}{2\pi n} + \\ &+ \frac{\lambda}{4!} \left[ (\omega_0 + \omega_n)^4 + (12\omega_0^2 + 24\omega_0 \omega_n + 14\omega_n^2) \frac{1}{4\pi n} + \right. \\ &\quad \left. + 4(\omega_0 + \omega_n) \sum_{k>0} \frac{\omega_k}{4\pi k} + \sum_{k>0} \frac{\omega_k^2}{4\pi k} \right] + \\ &+ \frac{m^2}{2} T + \frac{\lambda}{8} T^2. \end{aligned} \quad (4.13)$$

From the divergence structure above it is clear that all coefficients can be made finite by adding the counterterm

$$\delta H/2L = -\frac{\lambda}{4} T \sum_{n=1}^{\infty} \frac{1}{2\pi n} a_n^\dagger a_n - \frac{\lambda}{4} T \omega^2 - \frac{1}{2} m^2 T - \frac{\lambda}{8} T^2, \quad (4.14)$$

which can be obtained from (4.3) by integrating over  $x^-$  and decomposing the field. The renormalisation is thus standard, *i.e.* performed by normal-ordering and formally achieved by setting  $T = 0$  in the expressions above.

It is convenient to rescale the coefficients

$$\omega_0 \rightarrow \frac{\omega_0}{\sqrt{4\pi}} , \quad (4.15)$$

$$\omega_n \rightarrow \frac{\omega_n}{\sqrt{4\pi}} , \quad (4.16)$$

and define a dimensionless coupling  $g$  as

$$g \equiv \frac{\lambda}{24\pi m^2} , \quad (4.17)$$

such that (4.7) - (4.10) become

$$H_0/2L = \frac{m^2}{4\pi} \left[ \frac{1}{2} \omega_0^2 + \frac{g}{4} \left( \omega_0^4 + 4\omega_0 \sum_{n=1}^{\infty} \frac{\omega_n}{n} + \sum_{n=1}^{\infty} \frac{\omega_n^2}{n} \right) \right] , \quad (4.18)$$

$$\begin{aligned} H_n/2L &= \frac{m^2}{4\pi} \left[ \frac{1}{2} \left( \omega_n^2 + 2\omega_0 \omega_n + \frac{2}{n} \right) + \right. \\ &\quad \left. + \frac{g}{4} \left( (\omega_0 + \omega_n)^4 - \omega_0^4 + \frac{12}{n} (\omega_0 + \omega_n)^2 + 4\omega_n \sum_{k=1}^{\infty} \frac{\omega_k}{k} + 2\frac{\omega_n^2}{n} \right) \right] , \end{aligned} \quad (4.19)$$

and

$$\omega_0 + g \left( \omega_0^3 + \sum_{n=1}^{\infty} \frac{\omega_n}{n} \right) = 0 , \quad (4.20)$$

$$\omega_n + g \left( \omega_n^3 + 3\omega_0 \omega_n^2 + 3\omega_0^2 \omega_n + \frac{6}{n} (\omega_0 + \omega_n) \right) = 0 . \quad (4.21)$$

This system of equations has a trivial solution

$$\omega_0 = \omega_n = 0 , \quad (4.22)$$

corresponding to the symmetric phase with vanishing VEV of the field. The non-trivial solutions cannot be obtained exactly. If we assume that there is a critical coupling where the field starts to develop a VEV, then very close to this coupling the VEV  $\omega_0$  should be small and we expand  $\omega_n$  as a power series in  $\omega_0$ . This was already done in [18], and we merely quote the result (note the different normalization),

$$\begin{aligned} \omega_n &= -\frac{6g}{n+6g} \omega_0 + \frac{gn}{n+6g} \left( 1 - \left( \frac{n}{n+6g} \right)^3 \right) \omega_0^3 + O(\omega_0^5) \\ &\equiv \alpha_n(g) \omega_0 + \beta_n(g) \omega_0^3 + O(\omega_0^5) . \end{aligned} \quad (4.23)$$

Inserting this in (4.20) one obtains  $\omega_0$  as a function of the coupling  $g$  via

$$1 + g \sum_{n>0} \frac{\alpha_n}{n} + g \omega_0^2 \left( 1 + \sum_{n>0} \frac{\beta_n}{n} \right) = 0 . \quad (4.24)$$

As  $\beta_n > 0$ , this equation develops two real solutions with opposite sign if

$$1 + g \sum_{n>0} \frac{\alpha_n}{n} \leq 0 . \quad (4.25)$$

The critical coupling  $g_c$  is determined if equality holds. Using the explicit form of  $\alpha_n$  from (4.23), one finds

$$1 - g_c [\Psi(1 + 6g_c) + \gamma] = 0 , \quad (4.26)$$

with a numerical value of the critical coupling

$$g_c = 0.53070059 \dots . \quad (4.27)$$

which in terms of the original parameters is (*cf.* [28])

$$\lambda_c = 24\pi g_c m^2 \simeq 40.0 m^2 . \quad (4.28)$$

Above this coupling, the VEV  $\omega_0$  is non-vanishing and acquires one of two possible signs so that the reflection symmetry is spontaneously broken. Note again the different normalization of the coupling compared to [18],

$$g_{\text{old}} = 6 g_{\text{new}} . \quad (4.29)$$

The dependence of the rescaled VEV  $\omega_0$  is plotted in Fig.1. The critical exponent is  $1/2$ , implying mean-field behaviour as expected from the simple form of the ansatz (4.4). Also note the non-analyticity of  $\omega_0$  in the coupling at  $g_c$ .

From the one-particle energies (4.19) one can calculate the mass-shift induced by the non-perturbative ZM  $\omega$  as given by the ansatz (4.4),

$$\begin{aligned} \delta m^2 &= 2P_n^+ P_n^- - m^2 = 2 \frac{\pi n}{L} H_n - m^2 = m^2 \left[ \frac{n}{2} (\omega_n^2 + 2\omega_0 \omega_n) + \right. \\ &\quad \left. + \frac{ng}{4} \left( (\omega_0 + \omega_n)^4 - \omega_0^4 + \frac{12}{n} (\omega_0 + \omega_n)^2 + 4\omega_n \sum_{k=1}^{\infty} \frac{\omega_k}{k} + 2 \frac{\omega_n^2}{n} \right) \right] . \end{aligned} \quad (4.30)$$

Using the expression (4.23) for  $\omega_n$  this becomes (near the critical coupling)

$$\begin{aligned} \delta m^2 &= m^2 \omega_0^2 \left\{ 3g - 6g \left[ 1 - \frac{g}{1 + 6g/n} (\Psi(1 + 6g) + \gamma) \right] + \right. \\ &\quad \left. + 18g^2 \frac{n + 7g}{(n + 6g)^2} \right\} . \end{aligned} \quad (4.31)$$

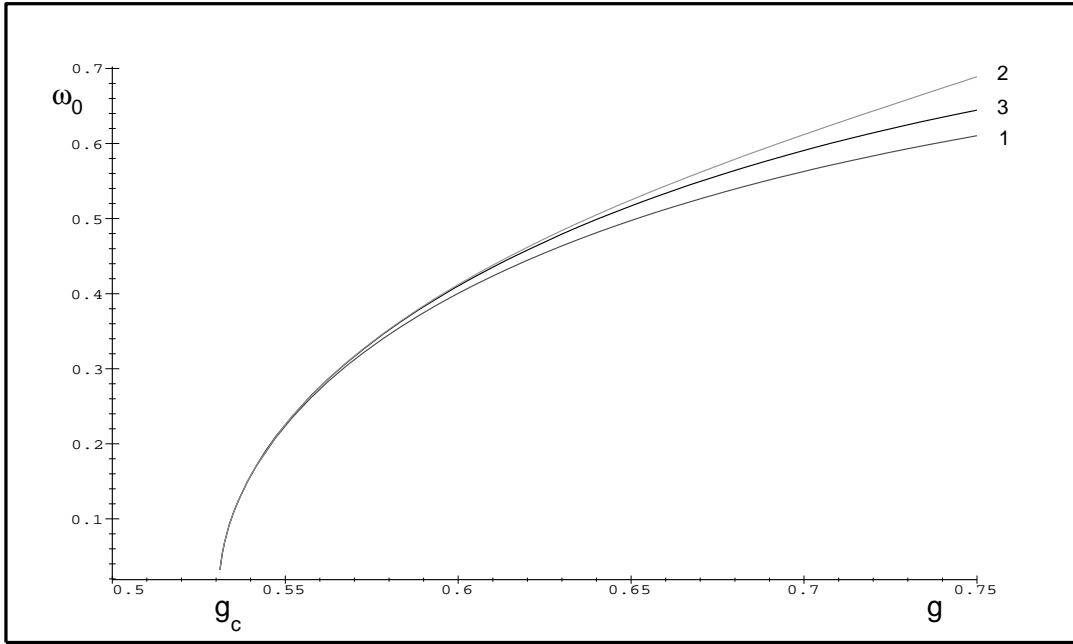


Figure 1: *Behaviour of the rescaled VEV  $\omega_0$  close to the critical coupling  $g_c$ . Graphs 1,2 and 3 refer to the expansion (4.23) of  $\omega_n$  in  $\omega_0$  up to third, fifth and seventh order, respectively.*

In contrast to the perturbative result (3.13), this expression is *not* vanishing in the continuum limit ( $n, L \rightarrow \infty$ ,  $n/L$  finite),

$$\delta m^2 \rightarrow m^2 \omega_0^2 \left\{ 3g - 6g \left[ 1 - g(\Psi(1 + 6g) + \gamma) \right] \right\}. \quad (4.32)$$

Due to (4.25) and (4.26) the term in the square brackets is a small (negative) quantity. The mass shift induced by  $\omega$  is thus positive. We want to emphasize that the ZM  $\omega$  has a non-trivial influence on the spectrum.

## 5 The Tamm-Dancoff Approximation

## 5.1 General Remarks

The Tamm-Dancoff approximation (TDA) was originally designed within the equal-time formulation of quantum field theory [39]. The core of the method was to enormously reduce the particle number to a finite (and small) one. Due to the complicated many-body structure of the equal-time vacuum, however, the approximation failed to lead to quantitative results and was abandoned thereafter. Although it has been noted rather early that LF field theory might be better suited for a TDA [40], it was not until recently that people began to start systematic investigations [41]. The hope was (and is), of course, that many of the problems of the original TDA can be avoided due to the simplicity of the LF vacuum. Specifically, people tried to calculate bound states for 1+1 dimensional field theories and were able to obtain reasonable results [42].

However, it became clear in the meantime, that there remain a number of open problems in connection with renormalisation, especially in more than 1+1 space-time dimensions [43, 44]. The structure of counterterms as well as their required number is unclear. Symmetries like rotational invariance are often violated [45]. In recent publications on LF  $\phi^4$ -theory with a TD truncation there even remained a logarithmic UV divergence, the origin of which was rather unclear [24].

In the following we will try to shed some light on the issue of renormalisation, in particular on the construction of the counterterms needed. These could, in principle, be different for different particle number sectors ("sector-dependent renormalisation" [41]). For a super-renormalisable theory like  $\phi_{1+1}^4$ , which conventionally is renormalisable by normal-ordering, this would be a rather undesired feature since it would complicate the renormalisation procedure enormously. Therefore, in the following, we will try to keep the renormalisation as simple as possible. To this end, we attempt to incorporate the normal-ordering prescription into the TDA. First we need some definitions. One- and two-particle states are given by

$$|n\rangle = a_n^\dagger |0\rangle \quad (5.1)$$

$$|m, n\rangle = a_m^\dagger a_n^\dagger |0\rangle \quad (5.2)$$

with normalization

$$\langle m|n\rangle = \delta_{mn} \quad (5.3)$$

$$\langle k, l|m, n\rangle = \delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm} . \quad (5.4)$$

The projection operators on the lowest particle number states are



$$P_0 = |0\rangle\langle 0| \quad (5.5)$$

$$P_1 = \sum_{n>0} |n\rangle\langle n| \quad (5.6)$$

$$P_2 = \frac{1}{2} \sum_{n,m>0} |n,m\rangle\langle m,n| \quad (5.7)$$

$$\vdots$$

$$P_N = \frac{1}{N!} \sum_{n_1,\dots,n_N>0} |n_1,\dots,n_N\rangle\langle n_1,\dots,n_N|. \quad (5.8)$$

Most important will be the Tamm-Dancoff projector

$$\mathbb{P}_N \equiv \sum_{\alpha=1}^N P_\alpha, \quad (5.9)$$

which projects onto the direct sum of all sectors with particle number less than or equal to  $N$ . Before we perform any detailed calculation, let us make a few remarks about the relativistic invariance of the TDA [46].

Let  $\mathcal{P}$  denote the Poincaré group. In  $d$  space-time dimensions the number of Poincaré generators is

$$\dim \mathcal{P} = \frac{d(d+1)}{2} \equiv D. \quad (5.10)$$

If the Poincaré generators are  $G_0, G_1, \dots, G_{D-1}$ , the TDA is relativistically invariant if

$$[\mathbb{P}_N G_i \mathbb{P}_N, \mathbb{P}_N G_k \mathbb{P}_N] = \mathbb{P}_N [G_i, G_k] \mathbb{P}_N. \quad (5.11)$$

It is easy to see, that this expression can only hold if at most one of the Poincaré generators fails to commute with  $\mathbb{P}_N$ . So we must have *e.g.*

$$[G_0, \mathbb{P}_N] \neq 0, \quad (5.12)$$

$$[G_k, \mathbb{P}_N] = 0, \quad k = 1, \dots, D-1. \quad (5.13)$$

The last identity implies that the  $G_k$  conserve particle number and therefore must be kinematical, *i.e.* interaction independent. Thus,  $G_0$  must be the only dynamical generator. Clearly, this can only happen if the dimension of the Poincaré group itself is small and if its stability group (of kinematical generators) is large. There is only one single case when the requirements (5.13) are met, namely light-front quantisation in  $d = 1+1$ , the case we are discussing in this paper! Here, according to (5.10), there are only three generators; the dynamical generator is  $G_0 \equiv P^-$ , the light-front energy; the kinematical generators

are the momentum  $P^+$  and the (longitudinal) boost  $M^{+-}$ . We think that this unique feature is one of the reasons, why the TDA (or more generally: Fock space truncation methods) work so well for LF field theories in  $d = 1+1$ .

Encouraged by this observations we continue and analyse the impact of the TDA on the quantum nature of our scalar field theory. This will be important for the issue of normal-ordering. We imagine that any operator  $\mathcal{O}$  can be built from the elementary Fock operators  $a_n, a_n^\dagger$ , no matter how complicated its form may be. Thus, we *define* the  $N$ -particle (NP) TDA by the replacement

$$a_n \rightarrow \mathbb{P}_N a_n \mathbb{P}_N, \quad (5.14)$$

$$a_n^\dagger \rightarrow \mathbb{P}_N a_n^\dagger \mathbb{P}_N. \quad (5.15)$$

In this way, however, one is mutilating the quantum structure of the theory. This can be seen by calculating the NPTDA of the elementary commutator

$$[a_m, a_n^\dagger]_N \equiv [\mathbb{P}_N a_m \mathbb{P}_N, \mathbb{P}_N a_n^\dagger \mathbb{P}_N]. \quad (5.16)$$

For arbitrary  $N$ , this commutator is not quite straightforwardly evaluated, so let us briefly go through the relevant steps. Firstly, we have

$$[a_m, a_n^\dagger]_N = \delta_{mn} \mathbb{P}_N + \mathbb{P}_N [a_m, \mathbb{P}_N] a_n^\dagger \mathbb{P}_N + \mathbb{P}_N a_n^\dagger [a_m, \mathbb{P}_N] \mathbb{P}_N, \quad (5.17)$$

so that we need

$$[a_n, \mathbb{P}_N] = (\mathbb{P}_{N-1} - \mathbb{P}_N) a_n = -P_N a_n, \quad (5.18)$$

which can be found inductively. Using the additional identity

$$\mathbb{P}_N P_N = P_N, \quad (5.19)$$

expression (5.17) becomes

$$[a_m, a_n^\dagger]_N = \delta_{mn} \mathbb{P}_{N-1} - P_N a_n^\dagger a_m \mathbb{P}_N - \mathbb{P}_N a_n^\dagger P_N a_m \mathbb{P}_N. \quad (5.20)$$

It is important to note that the operator  $a_n^\dagger P_N a_m$  has non-vanishing matrix elements in the  $N+1$ -particle sector only. The same is true for the operator  $P_N a_n^\dagger a_m$  in the  $N$ -particle sector. This leads to the final result

$$[a_m, a_n^\dagger]_N = \delta_{mn} \mathbb{P}_{N-1} - P_N a_n^\dagger a_m P_N. \quad (5.21)$$

Taking matrix elements of this expression one readily sees that the correct result ( $\delta_{mn}$ ) for the commutator within NPTDA is reproduced only up to the  $(N-1)$ -particle sector.

The additional term on the *r.h.s.* of (5.21) is acting in the  $N$ -particle subspace only. In other words, matrix elements of expressions involving the elementary commutator, which are calculated within NPTDA, should not be trusted beyond the  $(N-1)$ -particle sector. This will be relevant for the problem of normal-ordering to be discussed shortly.

To be a little bit more explicit, we list the lowest order expressions for (5.17)

$$[a_m, a_n^\dagger]_1 = \delta_{mn} |0\rangle\langle 0| - |n\rangle\langle m| , \quad (5.22)$$

$$[a_m, a_n^\dagger]_2 = \delta_{mn} \left[ |0\rangle\langle 0| + \sum_{l>0} |l\rangle\langle l| \right] - \sum_{l>0} |l, n\rangle\langle m, l| . \quad (5.23)$$

What are now the implications of all that for the renormalisation, in particular mass renormalisation? As can be seen from (4.1), the latter is encoded in the normal-ordering prescription of the expression

$$\begin{aligned} \frac{1}{2L} \int_{-L}^L dx^- \varphi^2(x) &= \frac{1}{2L} \int_{-L}^L dx^- : \varphi(x)^2 : + T \\ &= \sum_{n>0} \frac{1}{2\pi n} a_n^\dagger a_n + \sum_{n>0} \frac{1}{4\pi n} [a_n, a_n^\dagger] . \end{aligned} \quad (5.24)$$

Normal-ordering thus amounts to splitting off the divergent tadpole contribution  $T$ , which, in the Fock space language, is given by an elementary commutator (or contraction). Thus the remarks above, leading to (5.21), apply. Let us calculate the NPTDA of (5.24),

$$\begin{aligned} \frac{1}{2L} \int_{-L}^L dx^- \varphi^2(x) \stackrel{\text{NPTDA}}{\simeq} &= \sum_{n>0} \frac{1}{2\pi n} \mathbb{P}_{\mathbf{N-1}} a_n^\dagger \mathbb{P}_{\mathbf{N-1}} a_n \mathbb{P}_{\mathbf{N-1}} + \mathbb{P}_{\mathbf{N-1}} T - \\ &- \sum_{n>0} \frac{1}{4\pi n} P_N a_n^\dagger a_n P_N . \end{aligned} \quad (5.25)$$

The same remarks as for (5.21) are in order. Matrix elements of the last expression should not be expected to be consistent beyond the  $(N-1)$ -particle sector. Furthermore, one should note that, as  $\varphi^2$  is a one-body operator, there is only *one* commutator (or contraction) involved in the above normal-ordering. For a  $k$ -body operator we therefore conjecture that its renormalisation will be correct only up to the  $(N-k)$ -particle sector. For example, in order to get the renormalisation of the two-body operator  $\varphi^4$  correct up to the one-particle sector, a three-particle TDA will be needed.

## 5.2 One-Particle Light-Front Tamm-Dancoff Approximation

The one-particle LFTDA is defined by the replacements<sup>3</sup>

$$a_n \simeq \mathbb{P}_1 a_n \mathbb{P}_1 = |0\rangle\langle n| \quad (5.26)$$

$$a_n^\dagger \simeq \mathbb{P}_1 a_n^\dagger \mathbb{P}_1 = |n\rangle\langle 0| . \quad (5.27)$$

From our general results we expect that within one-particle LFTDA we will get a consistent renormalisation of the one-body operator  $\varphi^2$  in the vacuum sector only.

To solve for the ZM  $\omega$  we make the following TD ansatz

$$\omega_{TD} = c_0 |0\rangle\langle 0| + \sum_{n>0} c_n |n\rangle\langle n| . \quad (5.28)$$

We similarly expand the constraint  $\theta$  and the Hamiltonian  $H$

$$\theta \simeq \bar{\theta}_0 |0\rangle\langle 0| + \sum_{n>0} \bar{\theta}_n |n\rangle\langle n| , \quad (5.29)$$

$$H \simeq \bar{H}_0 |0\rangle\langle 0| + \sum_{n>0} \bar{H}_n |n\rangle\langle n| , \quad (5.30)$$

where the bars simply indicate the distinction of the coefficients above from those of the MF ansatz (4.5) and (4.6). The coefficients can be found as functions of  $c_0$ ,  $c_n$  upon inserting the ansatz (5.28) into the constraint (2.21) and the Hamiltonian (2.20) yielding

$$\bar{\theta}_0 = \left( m^2 + \frac{\lambda}{3} T \right) c_0 + \frac{\lambda}{3!} \left( c_0^3 + \sum_{n>0} \frac{c_n}{4\pi n} \right) , \quad (5.31)$$

$$\bar{\theta}_n = \left( m^2 + \frac{\lambda}{12\pi n} \right) c_n + \frac{\lambda}{3!} \left( c_n^3 + \frac{c_0}{4\pi n} \right) , \quad (5.32)$$

and

$$\frac{\bar{H}_0}{2L} = \frac{1}{2} \left( m^2 + \frac{\lambda}{4} T \right) c_0^2 + \frac{\lambda}{4!} \left( c_0^4 + 2 \sum_{n>0} \frac{c_0 c_n}{4\pi n} + \sum_{n>0} \frac{c_n^2}{4\pi n} \right) + \frac{m^2}{2} T + \frac{\lambda}{4!} T^2, \quad (5.33)$$

$$\frac{\bar{H}_n}{2L} = \frac{1}{2} \left( m^2 + \frac{\lambda}{16\pi n} \right) c_n^2 + \frac{1}{2} \left( m^2 + \frac{\lambda}{12} T \right) \frac{1}{4\pi n} + \frac{\lambda}{4!} \left( c_n^4 + \frac{c_0 c_n}{2\pi n} + \frac{c_0^2}{4\pi n} \right). \quad (5.34)$$

---

<sup>3</sup>Usually, the LFTDA refers only to the technique of solving bound state problems by using a few-body ansatz for the bound-state wave function[41, 42]. We use the term LFTDA in a wider context, defined essentially by the replacements (5.14) and (5.15).

At a first look, these expressions for the coefficients appear to be a disaster: the infinities in form of the tadpole  $T$  do not appear systematically, the mass gets renormalised in the vacuum sector only, but differently for the constraint  $\theta$  and the Hamiltonian  $H$ . Both expressions differ from the standard expression  $m^2 + \lambda T/2$ . There is also a divergent contribution from the  $\varphi^4$  term to  $\bar{H}_0$  which differs from the usual  $\lambda T^2/8$  of (4.1). As stated above, we do not believe the coefficients  $\bar{\theta}_n$  and  $\bar{H}_n$  to be correct within one-particle LFTDA. They will be discussed in the next subsection, when we go to higher order.

The way to remedy the situation (for the coefficients  $\bar{\theta}_0$  and  $\bar{H}_0$ ) is the following. We insist, firstly, on the standard mass renormalisation,  $m^2 + \lambda T/2$ , however, according to our general discussion, in the vacuum sector only. Secondly, we do not assume that the coefficients  $c_0$  and  $c_n$  are independent, and use this freedom to *redefine*  $c_n$  in the following way

$$c_0 \equiv \omega_0 , \quad (5.35)$$

$$c_n \equiv \omega_0 + \omega_n . \quad (5.36)$$

Inserting this into (5.31) and (5.33) one finds

$$\bar{\theta}_0 = \left( m^2 + \frac{\lambda}{2} T \right) \omega_0 + \frac{\lambda}{3!} \left( \omega_0^3 + \sum_{n>0} \frac{\omega_n}{4\pi n} \right) , \quad (5.37)$$

$$\begin{aligned} \bar{H}_0/2L &= \frac{1}{2} \left( m^2 + \frac{\lambda}{2} T \right) \omega_0^2 + \frac{\lambda}{4!} \left( \omega_0^4 + 4 \sum_{n>0} \frac{\omega_0 \omega_n}{4\pi n} + \sum_{n>0} \frac{\omega_n^2}{4\pi n} \right) + \\ &+ \frac{m^2}{2} T + \frac{\lambda}{4!} T^2 . \end{aligned} \quad (5.38)$$

Remarkably, the simple redefinition (5.36) has led to the desired results. The mass renormalisation is standard and the same for  $\theta_0$  and  $H_0$ . The divergences thus can be made to vanish by adding the counterterm (4.14). Both equations (5.37) and (5.38) coincide with the lowest order results from the mean-field ansatz (4.7) and (4.9) (up to the constant  $\varphi^4$ -contribution to  $H_0$  given by  $\lambda T^2/4!$ ). Note that there are no two-body ( $T^2$ ) contributions to the constraint. This is obviously true to all orders, so the renormalisation of  $\theta$  is slightly simpler than that of  $H$ , namely just mass renormalisation.

The coincidence with the MF results is not accidental. If one calculates the lowest order matrix elements of the MF ansatz (4.4), one finds

$$\omega_0 = \langle 0 | \omega_{MF} | 0 \rangle = \langle 0 | \omega_{TD} | 0 \rangle = c_0 , \quad (5.39)$$

$$\omega_0 + \omega_n = \langle n | \omega_{MF} | n \rangle = \langle n | \omega_{TD} | n \rangle = c_n . \quad (5.40)$$

Analogous relations hold for the matrix elements of the constraint and the Hamiltonian,

$$\langle 0|\theta|0\rangle = \theta_0 = \bar{\theta}_0, \quad (5.41)$$

$$\langle 0|H|0\rangle = H_0 = \bar{H}_0, \quad (5.42)$$

$$\langle n|\theta|n\rangle = \theta_0 + \theta_n = \bar{\theta}_n, \quad (5.43)$$

$$\langle n|H|n\rangle = H_0 + H_n = \bar{H}_n. \quad (5.44)$$

Thus, after the redefinition (5.36), the zero- and one-particle matrix elements of  $\omega$  calculated within the MF ansatz and TDA coincide. As the renormalisation within MF ansatz was conventional and straightforward, it is not too surprising that the behaviour of the redefined TDA under renormalisation gets improved. This will be another guideline in the following.

### 5.3 Two-Particle Light-Front Tamm-Dancoff Approximation

If we now go one step further and include also two-particle states via

$$a_n \simeq \mathbb{P}_2 a_n \mathbb{P}_2 = |0\rangle\langle n| + \sum_{m>0} |m\rangle\langle m, n| \quad (5.45)$$

$$a_n^\dagger \simeq \mathbb{P}_2 a_n^\dagger \mathbb{P}_2 = |n\rangle\langle 0| + \sum_{m>0} |n, m\rangle\langle m|, \quad (5.46)$$

we should further improve our renormalisation program. We expect a consistent renormalisation of  $\varphi^2$ -contributions in the vacuum- and one-particle sector, and of  $\varphi^4$ -contributions in the vacuum sector. The extended ansatz for  $\omega$  becomes

$$\begin{aligned} \omega_{TD} &= c_0 |0\rangle\langle 0| + \sum_{n>0} c_n |n\rangle\langle n| + \\ &+ \frac{1}{2} \sum_{m, n>0} c_{mn} |m, n\rangle\langle m+n| + \frac{1}{2} \sum_{m, n>0} c_{mn}^* |m+n\rangle\langle m, n| + \\ &+ \frac{1}{4} \sum_{k, l, m, n>0} \delta_{k+l, m+n} c_{klmn} |k, l\rangle\langle m, n|. \end{aligned} \quad (5.47)$$

It is now very plausible (though we cannot prove it a priori) that a consistent renormalisation requires a redefinition also of the coefficients  $c_{mn}$ ,  $c_{mn}^*$ , and  $c_{klmn}$ . To proceed as before, we would need the matrix elements  $\bar{\theta}_0$ ,  $\bar{\theta}_n$ ,  $\bar{H}_0$ , and  $\bar{H}_n$  with all two-particle contributions in order to just get a consistent renormalisation up to the one-particle sector.

This is very tedious and inefficient, as we are *only* interested in the *divergent* contributions from higher order terms to lower order matrix elements. Fortunately, there is an alternative: we simply demand that the matrix elements of  $\omega_{TD}$  and  $\omega_{MF}$  coincide also in the two-particle sector. This gives us the desired redefinitions, namely

$$c_{mn} = 2\omega_{mn} , \quad (5.48)$$

$$c_{mn}^* = 2\omega_{mn}^* , \quad (5.49)$$

$$c_{klmn} = (\omega_0 + \omega_n + \omega_n)(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) + 4\delta_{k+l,m+n}\omega_{klmn} . \quad (5.50)$$

Thus, essentially, only the two-particle matrix elements  $c_{klmn}$  get redefined. Expression (5.47) becomes

$$\begin{aligned} \omega_{TD} = & \omega_0|0\rangle\langle 0| + \sum_{n>0}(\omega_0 + \omega_n)|n\rangle\langle n| + \\ & + \sum_{m,n>0}\omega_{mn}|m,n\rangle\langle m+n| + \sum_{m,n>0}\omega_{mn}^*|m+n\rangle\langle m,n| + \\ & + \frac{1}{2}\sum_{m,n>0}(\omega_0 + \omega_m + \omega_n)|m,n\rangle\langle m,n| + \sum_{k,l,m,n>0}\delta_{k+l,m+n}\omega_{klmn}|k,l\rangle\langle m,n| . \end{aligned}$$

The diagonal two-particle term in the last line above contributes to the equations determining the coefficients  $\omega_0$  and  $\omega_n$  and crucially alters the renormalisation behaviour. In [24], it was noted that our MF ansatz amounts to including two-particle matrix elements, and it is just these terms that we have now explicitly displayed. Neglecting all terms containing  $\omega_{mn}$ ,  $\omega_{mn}^*$ , and  $\omega_{klmn}$ , which are not of interest within two-particle TDA, one finds for the constraint

$$\bar{\theta}_0 = \left(m^2 + \frac{\lambda}{2}T\right)\omega_0 + \frac{\lambda}{3!}\left(\omega_0^3 + \sum_{n>0}\frac{\omega_n}{4\pi n}\right) , \quad (5.51)$$

$$\bar{\theta}_n = \left(m^2 + \frac{\lambda}{2}T\right)(\omega_0 + \omega_n) + \frac{\lambda}{3!}\left[(\omega_0 + \omega_n)^3 + 6\frac{\omega_0 + \omega_n}{4\pi n} + \sum_{k>0}\frac{\omega_k}{4\pi k}\right] , \quad (5.52)$$

and for the Hamiltonian

$$\begin{aligned} \bar{H}_0/2L = & \frac{1}{2}\left(m^2 + \frac{\lambda}{2}T\right)\omega_0^2 + \frac{\lambda}{4!}\left[\omega_0^4 + 4\sum_{n>0}\frac{\omega_0\omega_n}{4\pi n} + \sum_{n>0}\frac{\omega_n^2}{4\pi n}\right] + \\ & + \frac{m^2}{2}T + \frac{\lambda}{8}T^2 , \\ \bar{H}_n/2L = & \frac{1}{2}\left(m^2 + \frac{\lambda}{2}T\right)(\omega_0 + \omega_n)^2 + \frac{1}{2}\left(m^2 + \frac{\lambda}{3}T\right)\frac{1}{2\pi n} + \end{aligned} \quad (5.53)$$

$$\begin{aligned}
& + \frac{\lambda}{4!} \left[ (\omega_0 + \omega_n)^4 + [12(\omega_0 + \omega_n)^2 + 2\omega_n^2] \frac{1}{4\pi n} + \right. \\
& \quad \left. + 4(\omega_0 + \omega_n) \sum_{k>0} \frac{\omega_k}{4\pi k} + \sum_{k>0} \frac{\omega_k^2}{4\pi k} \right] + \\
& + \frac{m^2}{2} T + \frac{\lambda}{4!} T^2 .
\end{aligned} \tag{5.54}$$

Several remarks are in order. As  $\theta$  does not contain two-body components, the renormalisation is correct up to the one-particle sector. (5.51) and (5.52) thus coincide with (4.7) and (4.12). The coefficient  $\theta_0$  is not even changed by including the two-particle contributions as can be seen by comparing with (5.37). In the vacuum coefficient  $\bar{H}_0$ , the mass renormalisation (due to the  $\varphi^2$  contributions) *and* the vacuum energy  $m^2 T/2 + \lambda T^2/8$  (with the  $T^2$  contribution stemming from the  $\varphi^4$ -term) are correct, as expected. So  $\bar{H}_0$  is consistently renormalised. In the one-particle coefficient  $\bar{H}_n$ , which should be compared with (4.13), only the mass renormalisation in the  $\omega$ -sector is correct, as this is due to one-body contributions like  $\omega^2 \varphi^2$ . As anticipated, mass renormalisation and vacuum energy stemming from the  $\varphi^4$ -term differ from the correct values by numerical factors. To get these correctly, one would have to perform a three-particle TDA. Presumably, this would only change the coefficients of the divergent terms, whereas the coefficients of the finite terms would remain the same. This would then be analogous to the change in  $\bar{H}_0$  by going from one-particle TDA (5.38) to two-particle TDA (5.53).

Summarizing, we can say that, in order to obtain a consistent renormalisation within a  $N$ -particle LFTD, one has to (i) include contributions from  $(N + 1)$ -particle matrix elements by (ii) appropriately redefining the coefficients in the TD ansatz. In this way, the Fock ansatz method (4.4) and the TDA become completely equivalent.

## 6 Discussion and Conclusion

In this paper we have reanalysed the vacuum structure of light-front  $\phi_{1+1}^4$ -theory by comparing different methods of solving for the constrained zero mode of the field operator. Within perturbation theory, the ZM induces a second order mass correction which is vanishing in the infinite volume limit. We believe that to all orders in perturbation theory the ZM only induces finite size effects, although we do not have a general proof.

We have presented two non-perturbative methods to obtain a solution for the ZM. An ansatz in terms of an increasing number of Fock operators, which we have truncated after the one-body term, seems to be the most economic procedure. With considerably



more efforts, exactly the same results can be obtained within a light-front Tamm-Dancoff approximation, if the renormalisation procedure is properly chosen. By doing so the logarithmic divergence of [24], stemming from an uncancelled tadpole term, does no longer appear<sup>4</sup>.

With either method we find a non-vanishing VEV  $\phi_c$  of the field if the coupling  $\lambda$  exceeds a critical value of  $\lambda_c \simeq 40m^2$ , implying spontaneous breakdown of the reflection symmetry  $\phi \rightarrow -\phi$ . As the VEV changes continuously, the associated phase transition is of second order, which has been rigorously established for the model at hand[47]. The order parameter  $\phi_c$  shows a square-root behaviour as a function of the coupling, so that the associated critical exponent is  $\beta = 1/2$ . The critical behaviour is thus of mean field type, which is wrong, as the  $\phi_{1+1}^4$  model is in the universality class of the two-dimensional Ising model; thus,  $\beta$  should be  $1/8$ . It is difficult to say, whether this shortcoming can be removed if one extends our approximations to higher orders. This question, of course, deserves further investigations.

Another problem we have to face is the absence of any volume dependence of the phase transition. We have been working in a finite spatial volume of length  $2L$ , the length scale  $L$ , however, drops out of the equation (4.26) determining the critical coupling. On the other hand, there cannot be a phase transition in a finite volume due to topological fluctuations (kinks and anti-kinks) which have non-vanishing statistical weight for  $L < \infty$ . We have not incorporated these fluctuations by our choice of (periodic) boundary conditions, so it is perhaps not too astonishing that we do not obtain a volume dependence. Work in this direction is underway.

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<sup>4</sup>The main difference between the cited work and ours is that in the former the authors do not restrict the particle number for intermediate states. This results in a divergence structure different from ours, and it seems to be rather difficult to find a systematic renormalisation procedure in this case.

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